

SCIENCE AND HUMANISM Physics in Our Time

ΒY

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TO MY COMPANION THROUGH THIRTY YEARS

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PREFACE

These are four public lectures which were delivered under the auspices of the Dublin Institute for Advanced Studies at University College Dublin in February, 1950 under the title 'Science as a Constituent of Humanism'. Neither this nor the abbreviated title chosen here adequately covers the whole, but rather the first sections only. In the remaining part, from p. 11 onward, I intend to depict the present situation in physics as it has gradually developed in the current century; to depict it from the point of view expressed in the title and in the earlier part, thus giving, as it were, an example of how I am looking on scientific effort: as forming part of man's endeavour to grasp the human situr ation.

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THE SPIRITUAL BEARING OF SCIENCE ON LIFE

What is the value of scientific research? Everybody knows that in our days more than ever before a man or a woman who wishes to make a genuine contribution to the advancement of science has to specialize: which means to intensify one's endeavour to learn all that is known within a certain narrow domain and then to try and increase this knowledge by one's own work-by studies, experiments, and thinking. Being engaged in such specialized activity one naturally at times stops to think what it is good for. Has the promotion of knowledge within a narrow domain any value in itself? Has the sum total of achievements in all the several branches of one science—say of physics, or chemistry, or botany, or zoology-any value in itself-or perhaps the sum total of the achievements of all the sciences together -and what value has it?

A great many people, particularly those not deeply interested in science, are inclined to answer this question by pointing to the practical consequences of scientific achievements in transforming technology, industry, engineering, etc., in fact in changing our whole way of life beyond recognition in the course

of less than two centuries, with further and even more rapid changes to be expected in the time to come.

Few scientists will agree with this utilitarian appraisal of their endeavour. Questions of values are, of course, the most delicate ones; it is hardly possible to offer incontrovertible arguments. But let me give you the three principal ones by which I should try to oppose this opinion.

Firstly, I consider natural science to be very much on the same line as the other kinds of learning—or Wissenschaft, to use the German expression—cultivated at our universities and other centres for the advancement of knowledge. Consider the study or research in history or languages, philosophy, geography—or history of music, painting, sculpture, architecture—or in archaeology and pre-history; nobody would like to associate with these activities, as their principal aim, the practical improvement of the conditions of human society, although improvement does result from them quite frequently. I cannot see that science has, in this respect, a different standing.

On the other hand (and this is my second argument), there are natural sciences which have obviously no practical bearing at all on the life of the human society: astrophysics, cosmology, and some branches of geophysics. Take, for instance, seismology. We know enough about earthquakes to know that there is very little chance of foretelling them, in the way of warning people to leave their houses, as we warn trawlers to return when a storm is

drawing near. All that seismology could do is to warn prospective settlers of certain danger zones; but those, I am afraid, are mostly known by sad experience without the aid of science, yet they are often densely populated, the need for fertile soil being more pressing.

Thirdly, I consider it extremely doubtful whether the happiness of the human race has been enhanced by the technical and industrial developments that followed in the wake of rapidly progressing natural science. I cannot here enter into details, and I will not speak of the future development—the surface of the earth getting infected with artificial radioactivity, with the gruesome consequences for our race, depicted by Aldous Huxley in his horribly interesting recent novel (Ape and Essence). But consider only the 'marvellous reduction of size' of the world by the fantastic modern means of traffic. All distances have been reduced to almost nothing, when measured not in miles but in hours of quickest transport. But when measured in the costs of even the cheapest transport they have been doubled or trebled even in the last 10 or 20 years. The result is that many families and groups of close friends have been scattered over the globe as never before. In many cases they are not rich enough ever to meet again, in others they do so under terrible sacrifices for a short time ending in a heart-rending farewell. Does that make for human happiness? These are a few striking examples; one could enlarge on the topic for hours.

But let us turn to less gloomy aspects of human activities. You may ask-you are bound to ask me now: What, then, is in your opinion the value of natural science? I answer: Its scope, aim and value is the same as that of any other branch of human knowledge. Nay, none of them alone, only the union of all of them, has any scope or value at all, and that is simply enough described: it is to obey the command of the Delphic deity, Γνῶθι σεαυτόν, get to know yourself. Or, to put it in the brief, impressive rhetoric of Plotinus (Enn. VI, 4, 14): ἡμεῖς δέ, τίνες δὲ ήμεις; 'And we, who are we anyhow?' He continues: 'Perhaps we were there already before this creation came into existence, human beings of another type, or even some sort of gods, pure souls and mind united with the whole universe, parts of the intelligible world, not separated and cut off, but at one with the whole.'

I am born into an environment—I know not whence I came nor whither I go nor who I am. This is my situation as yours, every single one of you. The fact that everyone always was in this same situation, and always will be, tells me nothing. Our burning question as to the whence and whither—all we can ourselves observe about it is the present environment. That is why we are eager to find out about it as much as we can. That is science, learning, knowledge, that is the true source of every spiritual endeavour of man. We try to find out as much as we can about the spatial and temporal surrounding of the place in which we find ourselves put by birth.

And as we try, we delight in it, we find it extremely interesting. (May that not be the end for which we are there?)

It seems plain and self-evident, yet it needs to be said: the isolated knowledge obtained by a group of specialists in a narrow field has in itself no value whatsoever, but only in its synthesis with all the rest of knowledge and only inasmuch as it really contributes in this synthesis something toward answering the demand Tives Sè ἡμεῖς; ('who are we?')

José Ortega y Gasset, the great Spanish philosopher, who is now after many years of exile back in Madrid (though he is, I believe, just as little a fascist as a sozialdemokrat, but just an ordinary reasonable person), published in the twenties of this century a series of articles, which were later collected in a delightful volume under the title of La rebelion de las masas—the rebellion of the masses. It has, by the way, nothing to do with social or other revolutions, the rebelión is meant purely metaphorically. The Age of Machinery has resulted in sending the numbers of the populations and the volume of their needs up to enormous heights, unprecedented and unforeseeable. The daily life of every one of us becomes more and more entangled with the necessity of coping with these numbers. Whatever we need or desire, a loaf of bread or a pound of butter, a bus-lift or a theatre-ticket, a quiet holiday resort or the permit to travel abroad, a room to live in or a job to live on . . . there are

always many, many others having the same need or desire. The new situations and developments that have turned up as the result of this unparalleled soaring of the numbers form the subject of Ortega's book.

It contains extremely interesting observations. Just to give you an example—though it does not concern us at the moment—one chapter-heading reads El major peligro, el estado: the greatest danger—the state. He there declares the increasing power of the state in curtailing individual freedom—under the pretext of protecting us, but far beyond necessity-to be the greatest danger to the future development of culture (kultur). But the chapter I wish to speak of here is the preceding one; it is entitled La barbarie del 'especialismo': the barbarism of specialization. At first sight it seems paradoxical and it may shock you. He makes bold to picture the specialized scientist as the typical representative of the brute ignorant rabble—the hombre masa (mass-man) who endanger the survival of true civilization. I can only pick out a few passages from the delightful description he gives of this 'type of scientist without precedent in history'.

He is a person who, of all the things that a truly educated person ought to know of, is familiar only with one particular science, nay even of this science only that small portion is known to him in which he himself is engaged in research. He reaches the point where he proclaims it a virtue not to take any notice of all that remains outside the narrow domain he himself cultivates, and denounces as dilettantist the curiosity that aims at the synthesis of all knowledge.

It comes to pass that he, secluded in the narrowness of his field of vision, actually succeeds in discovering new facts and in promoting his science (which he hardly knows) and promoting along with it the integrated human thought—which he with full determination ignores. How has anything like this been possible, and how does it continue to be possible? For we must strongly underline the inordinateness of this undeniable fact: experimental science has been advanced to a considerable extent by the work of fabulously mediocre and even less than mediocre persons.

I shall not continue the quotation, but I strongly recommend you to get hold of the book and continue for yourself. In the twenty-odd years that have passed since the first publication, I have noticed very promising traces of opposition to the deplorable state of affairs denounced by Ortega. Not that we can avoid specialization altogether; that is impossible if we want to get on. Yet the awareness that specialization is not a virtue but an unavoidable evil is gaining ground, the awareness that all specialized research has real value only in the context of the integrated totality of knowledge. The voices become fainter and fainter that accuse a man of dilettantism who dares to think and speak and write on topics that require more than the special training for which he is 'licensed' or 'qualified'. And any loud barking at such attempts comes from very special quarters of two types—either very scientific or very unscientific quarters—and the reasons for the barking are in both cases translucent.

In an article on 'The German Universities' (published on 11 December 1949 in *The Observer*)

Robert Birley, Headmaster of Eton, quoted some lines from the report of the Commission for University Reform in Germany—quoted them very emphatically, an emphasis that I fully endorse. The following is said in this report:

Each lecturer in a technical university should possess the following abilities:

(a) To see the limits of his subject matter. In his teaching to make the students aware of these limits, and to show them that beyond these limits forces come into play which are no longer entirely rational, but arise out of life and human society itself.

(b) To show in every subject the way that leads beyond its own narrow confines to broader horizons of its own. Etc.

I won't say that these formulations are peculiarly original, but who would expect originality of a committee or commission or board or that sort of thing?—mankind en masse is always very commonplace. Yet one is glad and thankful to find this sort of attitude prevailing. The only criticism—if it be a criticism—is that one can see no earthly reason why these demands should be restricted to the teachers at technical universities in Germany. I believe they apply to any teacher at any university, nay, at any school in the world; I should formulate the demand thus:

Never lose sight of the role your particular subject has within the great performance of the tragicomedy of human life; keep in touch with life—not so much with practical life as with the ideal background of life, which is ever so much more important; and, Keep life in touch with you. If you cannot

—in the long run—tell everyone what you have been doing, your doing has been worthless.

THE PRACTICAL ACHIEVEMENTS OF SCIENCE TENDING TO OBLITERATE ITS TRUE IMPORT

I regard the public lectures which the statute of the Institute prescribes for us to deliver every year as one of the means for establishing and keeping up this contact in our small domain. Indeed I consider this to be their exclusive scope. The task is not very easy. For one has to have some kind of background to start from, and, as you know, scientific education is fabulously neglected, not only in this or that country -though, indeed, in some more than in others. This is an evil that is inherited, passed on from generation to generation. The majority of educated persons are not interested in science, and are not aware that scientific knowledge forms part of the idealistic background of human life. Many believe -in their complete ignorance of what science really is—that it has mainly the ancillary task of inventing new machinery, or helping to invent it, for improving our conditions of life. They are prepared to leave this task to the specialists, as they leave the repairing of their pipes to the plumber. If persons with this outlook decide upon the curriculum of our children, the result is necessarily such as I have just described it.

There are, of course, historical reasons why this attitude still prevails. The bearing of science on the

idealistic background of life has always been greatapart perhaps from the Dark Ages, when science practically did not exist in Europe. But it must be confessed that there has been a lull also in more recent times, which could easily deceive one into under-rating the idealistic task of science. I place the lull about in the second half of the nineteenth century. This was a period of enormous explosionlike development of science, and along with it of a fabulous, explosion-like development of industry and engineering which had such a tremendous influence on the material features of human life that most people forgot any other connections. Nay, worse than that! The fabulous material development led to a materialistic outlook, allegedly derived from the new scientific discoveries. These occurrences have, I think, contributed to the deliberate neglect of science in many quarters during the half century that followed—the one that is just drawing to a close. For there always is a certain time-lag between the views held by learned men and the views held by the general public about the views of those learned men. I do not think that fifty years is an excessive estimate for the average length of that time-lag.

Be that as it may, the fifty years that have just gone by—the first half of the twentieth century—have seen a development of science in general, and of physics in particular, unsurpassed in transforming our Western outlook on what has often been called the Human Situation. I have little doubt that it

will take another fifty years or so before the educated section of the general public will have become aware of this change. Of course, I am not so much of an idealistic dreamer as to hope substantially to accelerate this process by a couple of public lectures. But, on the other hand, this process of assimilation is not automatic. We have to labour for it. In this labour I take my share, trusting that others will take theirs. It is part of our task in life.

A RADICAL CHANGE IN OUR IDEAS OF MATTER

We shall now, at last, come down to some special topics. What I have said hitherto may seem pretty long, if you consider it a mere introduction. But I hope it is of some interest in itself—and I could not avoid it. I had to make clear the situation. None of the new discoveries about which I may tell you is frightfully exciting in itself. What is exciting, novel, revolutionary, is the general attitude we are compelled to adopt on any attempt to synthesize them all.

Let us go in medias res. There is the problem of matter. What is matter? How are we to picture matter in our mind?

The first form of the question is ludicrous. (How should we say what matter is—or, if it comes to that, what electricity is—both being phenomena given to us once only?) The second form already betrays the whole change of attitude: matter is an image in our mind—mind is thus prior to matter (notwith-

standing the strange empirical dependence of my mental processes on the physical data of a certain portion of matter, viz. my brain).

During the second half of the nineteenth century matter seemed to be the permanent thing to which we could cling. There was a piece of matter that had never been created (as far as the physicist knew) and could never be destroyed! You could hold on to it and feel that it would not dwindle away under your fingers.

Moreover this matter, the physicist asserted, was with regard to its demeanour, its motion, subject to rigid laws—every bit of it was. It moved according to the forces which neighbouring parts of matter, according to their relative situations, exerted on it. You could *foretell* the behaviour, it was rigidly determined in all the future by the initial conditions.

This was all quite pleasing, anyhow in physical science, in so far as external inanimate matter comes into play. When applied to the matter that constitutes our own body or the bodies of our friends, or even that of our cat or our dog, a well-known difficulty arises with regard to the apparent freedom of living beings to move their limbs at their own will. We shall enter on this question later (see p. 58 ff.) At the moment I wish to try and explain the radical change in our ideas about matter that has taken place in the course of the last half-century. It came about gradually, inadvertently, without anybody aiming at such a change. We believed we moved still within the old 'materialistic' frame of ideas, when it turned out that we had left it.

Our conceptions of matter have turned out to be 'much less materialistic' than they were in the second half of the nineteenth century. They are still very imperfect, very hazy, they lack clearness in various respects; but this can be said, that matter has ceased to be the simple palpable coarse thing in space that you can follow as it moves along, every bit of it, and ascertain the precise laws governing its motion.

Matter is constituted of particles, separated by comparatively large distances; it is embedded in empty space. This notion goes back to Leucippus and Democritus, who lived in Abdera in the fifth century B.C. This conception of particles and empty space (ἄτομοι καὶ κενόν) is retained today (with a modification that is just the thing I wish to explain now)—and not only that, there is complete historical continuity; that is to say, whenever the idea was taken up again it was in full awareness of the fact that one was taking up the concepts of the ancient philosophers. Moreover it experienced the greatest thinkable triumphs in actual experiment, such as the ancient philosophers would hardly have hoped for in their boldest dreams. For instance, O. Stern succeeded in determining the distribution of velocities among the atoms in a jet of silver vapour by the simplest and most natural means, of which figure 1 gives a rough schematical sketch. The outer circle (carrying the letters A, B, C) represents the crosssection of a closed cylindrical box, exhausted to perfect vacuum. The point S marks the cross-section

of an incandescent silver wire, which extends along the axis of the cylinder and continually evaporates silver atoms, that fly along straight lines, roughly speaking, in radial directions. However, the cylindrical shield Sh (smaller circle), disposed concentrically around S, lets them pass only at the opening O,

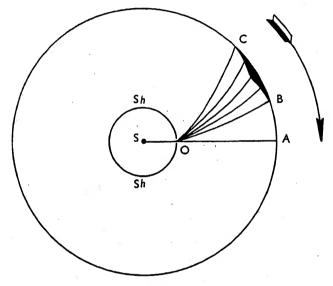


Figure 1

which represents a narrow slit parallel to the wire S. Without anything more, they pass on straight to A, where they are caught and, after a time, form a precipitate in the form of a narrow black line (parallel to the wire S and the slit O). But in Stern's experiment the whole apparatus is rotated, as on a potter's wheel, with high speed around the axis S (the sense

of the rotation shown by the arrow). This has the effect that the flying atoms-which are, of course, not affected by the rotation—are not precipitated at A but at points 'behind' A, the farther behind, the slower they are, because they allow the collecting surface to turn through a bigger angle before they reach it. Thus the slowest atoms form a line at C, the quickest at B. After a time one obtains a broad band whose cross-section is schematically indicated in our figure. By measuring its varying thickness and taking into account the dimensions of the apparatus and its speed of rotation, one can determine the actual velocity of the atoms, more particularly the relative numbers of atoms flying with various velocities—the so-called velocity distribution. I must still explain the fan-like spreading of the paths of the atoms and their curvature shown in the figure, both in apparent contradiction to what I said about the flying atoms not being affected by the rotation of the apparatus. I have taken the liberty to draw these lines though they are not the 'actual' paths of the atoms, but what their paths would appear to be to an observer sharing the rotation of the apparatus (just as we share the rotation of the earth). It is essential to make oneself clear that these 'relative paths' remain the same during the rotation. Hence we may continue the rotation as long as we please, for a substantial deposit to develop.

These important experiments confirmed quantitatively Maxwell's theory of gases, many years after this theory had been expounded. Today they have

been eclipsed, and all but forgotten, by far more impressive investigations.

The effect of a single fast particle can be observed as it impinges on a fluorescent screen and causes a faint flash of light, a scintillation. (If you have a watch with luminous figures, take it into a dark room and observe it with a moderately strong magnifying glass: you will then observe the scintillations caused by the impact of single He-ions, α-particles, as they are called in this context.) In a Wilson cloud chamber you can observe the paths of single particles, α-particles, electrons, mesons, . . . , their traces can be photographed and you can determine their curvature in a magnetic field; cosmic ray particles passing through a photographic emulsion produce nuclear disintegrations there, and both the primary and the secondary particles (if they are charged, as they usually are) trace their paths in the emulsion, so that the paths become visible when the plate is developed by the ordinary photographic procedure. I could give you more examples (but these will suffice) of the very direct way in which the old hypothesis of the particle structure of matter has been confirmed far beyond the keenest expectation of previous centuries.

Still less expected is the modification which our ideas about the nature of all these particles underwent during the same time—had to undergo willy-nilly—in consequence of other experiments and of theoretical considerations.

Democritus and all who followed on his path up

to the end of the nineteenth century, though they had never traced the effect of an individual atom (and probably did not hope ever to be able to), were yet convinced that the atoms are individuals, identifiable, small bodies just like the coarse palpable objects in our environment. It seems almost ludicrous that precisely in the same years or decades which let us succeed in tracing single, individual atoms and particles, and that in various ways, we have yet been compelled to dismiss the idea that such a particle is an individual entity which in principle retains its 'sameness' for ever. Quite the contrary, we are now obliged to assert that the ultimate constituents of matter have no 'sameness' at all. When you observe a particle of a certain type, say an electron, now and here, this is to be regarded in principle as an isolated event. Even if you do observe a similar particle a very short time later at a spot very near to the first, and even if you have every reason to assume a causal connection between the first and the second observation, there is no true, unambiguous meaning in the assertion that it is the same particle you have observed in the two cases. The circumstances may be such that they render it highly convenient and desirable to express oneself so, but it is only an abbreviation of speech; for there are other cases where the 'sameness' becomes entirely meaningless; and there is no sharp boundary, no clear-cut distinction between them, there is a gradual transition over intermediate cases. And I beg to emphasize this and I beg you to believe it: It is not a question of our

being able to ascertain the identity in some instances and not being able to do so in others. It is beyond doubt that the question of 'sameness', of identity, really and truly has no meaning.

FORM, NOT SUBSTANCE, THE FUNDAMENTAL CONCEPT

The situation is rather disconcerting. You will ask: What are these particles then, if they are not individuals? And you may point to another kind of gradual transition, namely that between an ultimate particle and a palpable body in our environment, to which we do attribute individual sameness. A number of particles constitute an atom. Several atoms go to compose a molecule. Molecules there are of various sizes, small ones and big ones, but without there being any limit beyond which we call it a big molecule. In fact there is no upper limit to the size of a molecule, it may contain hundreds of thousands of atoms. It may be a virus or a gene, visible under the microscope. Finally we may observe that any palpable object in our environment is composed of molecules, which are composed of atoms, which are composed of ultimate particles . . . and if the latter lack individuality, how does, say, my wrist-watch come by individuality? Where is the limit? How does individuality arise at all in objects composed of non-individuals?

It is useful to consider this question in some detail, for it will give us the clue to what a particle or an

atom really is—what there is permanent in it in spite of its lack of individuality. On my writing-table at home I have an iron letter-weight in the shape of a Great Dane, lying with his paws crossed in front of him. I have known it for many years. I saw it on my father's writing-desk when my nose would hardly reach up to it. Many years later, when my father died, I took the Great Dane, because I liked it, and I used it. It accompanied me to many places, until it stayed behind in Graz in 1938, when I had to leave in something of a hurry. But a friend of mine knew that I liked it so she took it and kept it for me. And three years ago, when my wife visited Austria, she brought it to me, and there it is again on my desk.

I am quite sure it is the same dog, the dog that I first saw more than fifty years ago on my father's desk. But why am I sure of it? That is quite obvious. It is clearly the peculiar form or shape (German: Gestalt) that raises the identity beyond doubt, not the material content. Had the material been melted and cast into the shape of a man, the identity would be much more difficult to establish. And what is more: even if the material identity were established beyond doubt, it would be of very restricted interest. I should probably not care very much about the identity or not of that mass of iron, and should declare that my souvenir had been destroyed.

I consider this a good analogy, and perhaps more than an analogy, for pointing out what the particles or atoms really are. For we can see in this example

as in many others how in palpable bodies, composed of many atoms, individuality arises out of the structure of their composition, out of shape or form, or organization, as we might call it in other cases. The identity of the material, if there is any, plays a subordinate role. You may see this particularly well in cases when you speak of 'sameness' though the material has definitely changed. A man returns after twenty years of absence to the cottage where he spent his childhood. He is profoundly moved by finding the place unchanged. The same little stream flows through the same meadows, with the cornflowers and poppies and willow trees he knew so well, the white-and-brown cows and the ducks on the pond, as before, and the collie dog coming forth with a friendly bark and wagging his tail to him. And so on. The shape and the organization of the whole place have remained the same, in spite of the entire 'change of material' in many of the items mentioned, including, by the way, our traveller's own bodily self! Indeed, the body he wore as a child has in the most literal sense 'gone with the wind'. Gone, and yet not gone. For, if I am allowed to continue my novelistic snapshot, our traveller will now settle down, marry, and have a small son, who is the very image of his father as old photographs show him at the same tender age.

Let us now return to our ultimate particles and to small organisations of particles as atoms or small molecules. The *old* idea about them was that *their* individuality was based on the identity of matter in

them. This seems to be a gratuitous and almost mystical addition that is in sharp contrast to what we have just found to constitute the individuality of macroscopic bodies, which is quite independent of such a crude materialistic hypothesis and does not need its support. The new idea is that what is permanent in these ultimate particles or small aggregates is their shape and organization. The habit of everyday language deceives us and seems to require, whenever we hear the word 'shape' or 'form' pronounced, that it must be the shape or form of something. that a material substratum is required to take on a shape. Scientifically this habit goes back to Aristotle, his causa materialis and causa formalis. But when you come to the ultimate particles constituting matter, there seems to be no point in thinking of them again as consisting of some material. They are, as it were, pure shape, nothing but shape; what turns up again and again in successive observations is this shape, not an individual speck of material.

THE NATURE OF OUR 'MODELS'

In this we must, of course, take shape (or Gestalt) in a much wider sense than as geometrical shape. Indeed there is no observation concerned with the geometrical shape of a particle or even of an atom. It is true that in thinking about the atom, in drafting theories to meet the observed facts, we do very often draw geometrical pictures on the black-board, or on a piece of paper, or more often just only in our mind,

the details of the picture being given by a mathematical formula with much greater precision and in a much handier fashion than pencil or pen could ever give. That is true. But the geometrical shapes displayed in these pictures are not anything that could be directly observed in the real atoms. The pictures are only a mental help, a tool of thought, an intermediary means, from which to deduce, out of the results of experiments that have been made, a reasonable expectation about the results of new experiments that we are planning. We plan them for the purpose of seeing whether they confirm the expectations—thus whether the expectations were reasonable, and thus whether the pictures or models we use are adequate. Notice that we prefer to say adequate, not true. For in order that a description be capable of being true, it must be capable of being compared directly with actual facts. That is usually not the case with our models.

But we do use them, as I said, to deduce observable features from them. It is these that constitute the permanent shape or form or organization of the material object, and they have usually nothing to do with 'tiny specks of material, constituting the object'.

Take for instance the atom of iron. A very interesting and highly complicated part of its organization can be displayed again and again, whenever you like and with unalterable permanence, in the following manner. You bring a small amount of iron (or of an iron salt) into the electric arc and take a photograph of its spectrum, produced by a powerful optical

grating. You find tens of thousands of sharp spectral lines, that is to say tens of thousands of definite wave-lengths contained in the light that an iron atom emits at these high temperatures. And they are always the same, exactly the same, so much so that as is well known, you can tell from the spectrum of a star that it contains certain chemical elements. While you are unable to find out anything about the geometrical shape of an atom—even with the most powerful microscope—you are able to discover its typical permanent organization, displayed in its spectrum, at a distance of thousands of light-years!

You may say the typical line spectrum of an element like iron is a macroscopic property, a property of the glowing vapour, it has nothing to do with its coarse-grained structure (its being composed of single atoms)—and nobody has yet observed the light emitted by a single, a truly isolated, atom. That is true. But, of course, I must remind you that the theory of matter, as it is accepted at present, does ascribe the emission of all these various monochromatic beams of light to the single atom; the geometricalmechanical-electrical constitution of the single atom is deemed responsible for every single wave-length we observe in the glowing vapour. To confirm this, the physicist most emphatically points to the fact that these line spectra are only observed in the rarefied vaporous state where the atoms are so far apart from each other that they do not disturb each other. Glowing solid or liquid iron emits a continuous spectrum, much the same as every other solid or liquid

at the same temperature—the sharp lines have entirely disappeared, or, better, they are entirely blurred, owing to the mutual disturbance of neighbouring atoms.

Would you then say—so you might ask me—would you then say that we are to regard the observed line spectra (which, broadly speaking, conform to the theory) as part of the *circumstantial evidence*, that the iron atoms of our theoretical description actually exist and that they constitute the vapour in the way the theory of gases maintains it—small specks of matter (of that peculiar constitution that makes them emit the spectral lines)—small specks of something, wide apart, embedded in the nothing, flying hither and thither, occasionally colliding with the walls, etc., etc.? Is that a true picture of glowing iron vapour?

I keep to what I said earlier in a more general context: it is certainly an adequate picture; but as regards its truth the appropriate question to ask is not whether it is true or not, but whether it is at all capable of being either true or false. Probably it is not. Probably we cannot ask for more than just adequate pictures capable of synthesizing in a comprehensible way all observed facts and giving a reasonable expectation on new ones we are out for.

Very similar declarations have been made again and again by competent physicists a long time ago, all through the nineteenth century and in the early days of our own. They were aware that the desire for having a *clear* picture necessarily led one to encumber it with unwarranted details. It is, so to speak,

'infinitely improbable' that those gratuitous additions should, by good luck, turn out to be 'correct'. L. Boltzmann strongly emphasized the point; let me be quite precise, he would say, childishly precise about my model, even though I know that I cannot guess from the ever incomplete circumstantial evidence of experiments what nature really is like. But without an absolutely precise model thinking itself becomes imprecise, and the consequences to be derived from the model become ambiguous.

Yet the attitude at that time—except perhaps in a very few philosophically foremost minds—was different from what it is now, it was still a little too naïve. While asserting that any model we may conceive is sure to be deficient and would surely be modified sooner or later, one still had at the back of one's mind the thought that a true model exists—exists so to speak in the Platonic realm of ideas—that we approach to it gradually, without perhaps ever reaching it, owing to human imperfections.

This attitude has now been abandoned. The failures we have experienced no longer refer to details, they are of a more general kind. We have become fully aware of a situation that may perhaps be summarized as follows. As our mental eye penetrates into smaller and smaller distances and shorter and shorter times, we find nature behaving so entirely differently from what we observe in visible and palpable bodies of our surrounding that no model shaped after our large-scale experiences can ever be 'true'. A completely satisfactory model of this type is not only practically

inaccessible, but not even thinkable. Or, to be precise, we can, of course, think it, but however we think it, it is wrong; not perhaps quite as meaningless as a 'triangular circle', but much more so than a 'winged lion'.

CONTINUOUS DESCRIPTION AND

I shall try to be a little clearer about this. From our experiences on a large scale, from our notion of

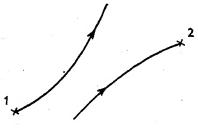


Figure 2

geometry and of mechanics—particularly the mechanics of the celestial bodies—physicists had distilled the one clear-cut demand that a truly clear and complete description of any physical happening has to fulfil: it ought to inform you precisely of what happens at any point in space at any moment of time—of course, within the spatial domain and the period of time covered by the physical events you wish to describe. We may call this demand the postulate of continuity of the description. It is this

postulate of continuity that appears to be unfulfillable! There are, as it were, gaps in our picture.

This is intimately connected with what I called earlier the lack of individuality of a particle, or even of an atom. If I observe a particle here and now, and observe a similar one a moment later at a place very near the former place, not only cannot I be sure whether it is 'the same', but this statement has no absolute meaning. This seems to be absurd. For we are so used to thinking that at every moment between the two observations the first particle must have been somewhere, it must have followed a path, whether we know it or not. And similarly the second particle must have come from somewhere, it must have been somewhere at the moment of our first observation. So in principle it must be decided, or decidable, whether these two paths are the same or not-and thus whether it is the same particle. In other words we assume—following a habit of thought that applies to palpable objects—that we could have kept our particle under continuous observation, thereby ascertaining its identity.

This habit of thought we must dismiss. We must not admit the possibility of continuous observation. Observations are to be regarded as discrete, disconnected events. Between them there are gaps which we cannot fill in. There are cases where we should upset everything if we admitted the possibility of continuous observation. That is why I said it is better to regard a particle not as a permanent entity but as an instantaneous event. Sometimes these events

form chains that give the illusion of permanent beings—but only in particular circumstances and only for an extremely short period of time in every single case.

Let us go back to the more general statement I made before, namely that the classical physicist's naïve ideal cannot be fulfilled, his demand that in principle information about every point in space at every moment of time should at least be thinkable. That this ideal breaks down has a very momentous consequence. For in the times when this ideal

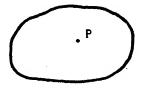


Figure 3

of continuity of description was not doubted, the physicists had used it to formulate the principle of causality for the purposes of their science in a very clear and precise fashion—the only one in which they could use it, the ordinary enouncements being much too ambiguous and imprecise. It includes in this form, the principle of 'close action' (or the absence of actio in distans) and runs as follows: The exact physical situation at any point P at a given moment t is unambiguously determined by the exact physical situation within a certain surrounding of P at any previous time, say t—τ. If τ is large, that is, if that previous time lies far back, it may be necessary to

know the previous situation for a wide domain around P. But the 'domain of influence' becomes smaller and smaller as τ becomes smaller, and becomes infinitesimal as τ becomes infinitesimal. Or, in plain, though less precise, words: what happens anywhere at a given moment depends only and unambiguously on what has been going on in the immediate neighbourhood 'just a moment earlier'. Classical physics rested entirely on this principle. The mathematical instrument to implement it was in all cases a system of partial differential equations—so-called field equations.

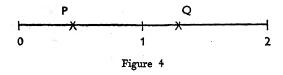
Obviously, if the ideal of continuous, 'gap-less', description breaks down, this precise formulation of the principle of causality breaks down. And we must not be astonished to meet in this order of ideas with new, unprecedented difficulties as regards causation. We even meet (as you know) with the statement that there are gaps or flaws in strict causation. Whether this is the last word or not it is difficult to say. Some people believe that the question is by no manner of means settled (among them, by the way, is Albert Einstein). I shall tell you a little later about the 'emergency exit', used at present to escape from the delicate situation. For the moment I wish to attach some further remarks to the classical ideal of continuous description.

THE INTRICACY OF THE CONTINUUM

However painful its loss may be, by losing it we probably lose something that is very well worth

losing. It seems simple to us, because the idea of the continuum seems simple to us. We have somehow lost sight of the difficulties it implies. That is due to a suitable conditioning in early childhood. Such an idea as 'all the numbers between 0 and 1' or 'all the numbers between 1 and 2' has become quite familiar to us. We just think of them geometrically as the distance of any point like P or Q from 0 (see fig. 4).

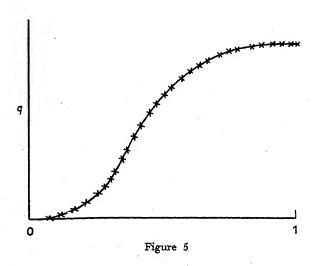
Among the points like Q there is also the $\sqrt{2}$ (= 1.414...). We are told that such a number as $\sqrt{2}$ worried Pythagoras and his school almost to exhaustion. Being used to such queer numbers from



early childhood, we must be careful not to form a low idea of the mathematical intuition of these ancient sages. Their worry was highly creditable. They were aware of the fact that no fraction can be indicated of which the square is exactly 2. You can indicate close approximations, as for instance $\frac{17}{12}$, whose square, $\frac{289}{144}$, is very near to $\frac{288}{144}$ which is 2. You can get closer by contemplating fractions with larger numbers than 17 and 12, but you will never get exactly 2.

The idea of a continuous range, so familiar to mathematicians in our days, is something quite

exorbitant, an enormous extrapolation of what is really accessible to us. The idea that you should really indicate the exact values of any physical quantity—temperature, density, potential, field strength, or whatever it might be—for all the points of a continuous range, say between zero and 1, is a bold



extrapolation. We never do anything else than determine the quantity approximately for a very limited number of points and then 'draw a smooth curve through them'. This serves us well for many practical purposes, but from the epistemological point of view, from the point of view of the theory of knowledge, it is totally different from a supposed exact continual description. I might add that even in classical physics there were quantities—as, for instance, temperature

or density—which avowedly did not admit of an exact continuous description. But this was due to the conception these terms represent—they have, even in classical physics, only a statistical meaning. However I shall not go into details about this at the moment, it would create confusion.

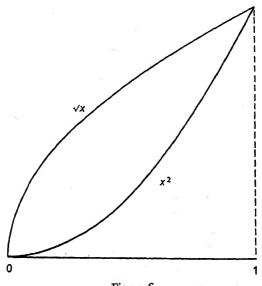


Figure 6

The demand for continuous description was encouraged by the fact that the mathematician claims to be able to indicate simple continuous descriptions of some of his simple mental constructions. For example, take again the range $0 \rightarrow 1$, call the variable in this range x, we claim to have an unambiguous idea of, say x^2 or \sqrt{x} .

The curves are pieces of parabolas (mirror images of each other). We claim to have full knowledge of every point of such a curve, or rather, given the horizontal distance (abscissa) we are able to indicate the height (ordinate) with any required precision. But behold the words 'given' and 'with any required precision'. The first means 'we can give the answer, when it comes to it'—we cannot possibly have all the answers in store for you in advance. The second means 'even so, we cannot as a rule give you an absolutely precise answer.' You must tell us the precision you require, e.g. up to 1000 decimal places.

Then we can give you the answer—if you leave us time.

Physical dependences can always be approximated by this simple kind of functions (the mathematician calls them 'analytical', which means something like 'they can be analysed'). But to assume that physical dependence is of this simple type, is a bold epistemological step, and probably an inadmissible step.

However, the chief conceptual difficulty is the enormous number of 'answers' that are required, due to the enormous number of points contained in even the smallest continuous range. This quantity—the number of points between 0 and 1, for example—is so fabulously great that it is hardly diminished

even if you take 'nearly all of them' away. Allow me to illustrate this by an impressive example.

Envisage again the line $0\rightarrow 1$. I wish to describe a certain set of points that is *left over*, when you take some of them away, bar them, exclude them, make them inaccessible—or whatever you wish to call it. I shall use the word 'take away'.

First take away the whole middle third including its left border point, thus the points from $\frac{1}{3}$ to $\frac{2}{3}$ (but you leave $\frac{2}{3}$). Of the remaining two thirds you again take away the 'middle thirds', including their left border points, but leaving their right border points. With the remaining 'four ninths' you proceed in the same way. And so on.

If you actually try to continue for only a few steps you will soon get the impression that 'nothing is left over'. Indeed at every step we take away a third of the remaining length. Now supposing the Income Tax Inspector charged you first 6s. 8d. in the £, and of the remainder again 6s. 8d. in the £, and so on, ad infinitum, you agree you would not retain much.

We shall now analyse our case, and you will be astonished how many of our numbers or points are left. I regret that this needs a little preparation. A number between zero and one can be represented by a decadic fraction, as

and you know this means

$$\frac{4}{10} + \frac{7}{10^2} + \frac{0}{10^3} + \frac{8}{10^4} + \dots$$

That we habitually use here the number 10 is a pure

accident, due to the fact that we have 10 fingers. We can use any other number, 8, 12, 3, 2.... We need, of course, different figure-symbols for all the numbers up to the chosen 'basis'. In our decadic system we need ten, 0, 1, 2, ... 9. If we used 12 as our basis, we should have to invent single symbols for 10 and 11. If we used the basis 8, the symbols for 8 and 9 would become supernumerary.

Non-decadic fractions have not altogether been ousted by the decimal system. Dyadic fractions, that is those which use the basis 2, are quite popular, particularly with the British. When I asked my tailor the other day how much material I should get him for the flannel trousers I had just ordered, he answered—to my amazement—1\frac{3}{8} yards. This is easily seen to be the dyadic fraction

1.011.

meaning

$$1 + \frac{0}{2} + \frac{1}{4} + \frac{1}{8}$$
.

In the same way some stock exchanges quote shares not in shillings and pence but in dyadic fractions of a pound, for example \pounds_{16}^{13} , which in *dyadic* notation would read

0.1101,

meaning

$$\frac{1}{2} + \frac{1}{4} + \frac{0}{8} + \frac{1}{16}$$
.

Notice that in a dyadic fraction only two symbols, viz. 0 and 1, occur.

For our present purpose we first need triadic

fractions, which have the basis 3 and use only the symbols 0, 1, 2. Here, for instance, the notation 0.2012...

means

$$\frac{2}{3} + \frac{0}{9} + \frac{1}{27} + \frac{2}{81} + \dots$$

(By adding dots we intentionally admit fractions that run to infinity, as for example the square root of 2). Now let us return to the problem of describing the 'almost vanishing' set of numbers that is left over in the construction illustrated by our figure. A little careful thinking will shew you that the points we have taken away are all those which in triadic representation contain a figure 1 somewhere. Indeed, by first cutting out the middle third we cut out all the numbers whose triadic fraction begins thus:

At the second step we cut out all those whose triadic fraction begins

And so on.—This consideration shews that there is something left, namely all those whose triadic fractions contain no 1, but only 0 and 2, as for instance

(where the dots stand for any sequence of 0s and 2s only). Among them are, of course, the *right* border points (as $0.2 = \frac{2}{3}$ or $0.22 = \frac{2}{3} + \frac{2}{9} = \frac{8}{9}$) of the excluded intervals; we had decided to let those border points stand. But there are a lot more, for instance the *periodic* dyadic fraction 0.20, meaning

0-20202020 ad infinitum. This is the infinite series

$$\frac{2}{3} + \frac{2}{3^3} + \frac{2}{3^5} + \frac{2}{3^7} + \dots$$

To find its value, think you multiply it by the square of 3, which is 9. Then the first term gives $\frac{18}{3}$, that is, 6, while the remaining terms give the same series again. Hence eight times our series is 6, and our number is $\frac{6}{3}$ or $\frac{3}{4}$.

Still, recalling again that the intervals we have Laken away' tend to cover the whole interval between o and 1, one is inclined to think that, compared with the original set (containing all numbers between 0 and 1), the remaining set must be 'exceedingly scarce'. But now comes the amazing turn: in a certain sense the remaining set is still just as vast as the original one. Indeed we can associate their respective members in pairs, by monogamously mating, as it were, each number of the original set with a definite number of the remaining set, without any number being left over on either side (the mathematician calls this a 'one-to-one correspondence'). This is so perplexing that, I am sure, many a reader will at first think he must have misunderstood the words, though I have taken pains to set them as unambiguously as possible.

How is this done? Well, the 'remaining set' is represented by all the triadic fractions containing only 0s and 2s; we gave the general example

0.22000202...

(the dots standing for any sequence of 0s and 2s

only). Associate with this triadic fraction the dyadic fraction

0.11000101...

obtained from the former by replacing every figure 2 by the figure 1. Vice versa you can, from any dyadic fraction, by changing its 1s into 2s, obtain the triadic representation of a definite number in what we called 'the remaining set'. Since now any member of the original set, that is, any number between 0 and 1, is represented by one and only one definite dyadic fraction, there is actually a perfect one-to-one mating between the members of the two sets.

[It may be useful to illustrate the 'mating' by examples. For instance the dyadic number that my tailor used

$$\frac{3}{8} = \frac{0}{2} + \frac{1}{4} + \frac{1}{8} = 0.011$$

would lead to the triadic counterpart

$$0.022 = \frac{0}{3} + \frac{2}{9} + \frac{2}{27} = \frac{8}{27};$$

that is to say, $\frac{3}{8}$ of the original set corresponds to $\frac{3}{27}$ in the remaining set. Inversely, take our triadic 0.20, meaning, as we made out, $\frac{3}{4}$. The corresponding dyadic 0.10 means the infinite series

$$\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + \frac{1}{2^9} + \dots$$

If you multiply this by the square of 2, which is 4, you get: 2+ the same series. In other words, three

¹ We have tacitly disregarded such trivial duplications as are instanced, in the decimal system, by 0.1 = 0.09 or 0.8 = 0.79.

times our series equals 2, the series equals $\frac{2}{3}$; that is to say, the number $\frac{3}{4}$ of the 'remaining set' corresponds (or 'is mated') to the number $\frac{2}{3}$ in the original set.

The remarkable fact about our 'remaining set' is that, though it covers no measurable interval, yet it has still the vast extension of any continuous range. This astonishing combination of properties is, in mathematical language, expressed by saying that our set has still the 'potency' of the continuum, although it is 'of measure zero'.

I have brought this case before you, in order to make you feel that there is something mysterious about the continuum and that we must not be all too astonished at the apparent failure of our attempts to use it for a precise description of nature.

THE MAKESHIFT OF WAVE MECHANICS

Now I shall try to give you an idea of the way in which physicists at present endeavour to overcome this failure. One might term it an 'emergency exit', though it was not intended as such, but as a new theory. I mean, of course, wave mechanics. (Eddington called it 'not a physical theory but a dodge—and a very good dodge too'.)

The situation is about as follows. The observed facts (about particles and light and all sorts of radiation and their mutual interaction) appear to be repugnant to the classical ideal of a continuous description in space and time. (Let me explain

myself to the physicist by hinting at one example: Bohr's famous theory of spectral lines in 1913 had to assume that the atom makes a sudden transition from one state into another state, and that in doing so it emits a train of light waves several feet long, containing hundreds of thousands of waves and requiring for its formation a considerable time. No information about the atom during this transition can be offered.)

So the facts of observation are irreconcilable with a continuous description in space and time; it just seems impossible, at least in many cases. On the other hand, from an incomplete description—from a picture with gaps in space and time—one cannot draw clear and unambiguous conclusions; it leads to hazy, arbitrary, unclear thinking—and that is the thing we must avoid at all costs! What is to be done? The method adopted at present may seem amazing to you. It amounts to this: we do give a complete description, continuous in space and time without leaving any gaps, conforming to the classical ideala description of something. But we do not claim that this 'something' is the observed or observable facts; and still less do we claim that we thus describe what nature (matter, radiation, etc.) really is. In fact we use this picture (the so-called wave picture) in full knowledge that it is neither.

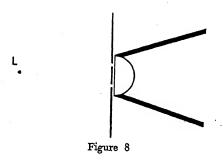
There is no gap in this picture of wave mechanics, also no gap as regards causation. The wave picture conforms with the classical demand for complete determinism, the mathematical method used is that

of field-equations, though sometimes they are a highly generalized type of field-equations.

But what is the use of such a description, which, as I said, is not believed to describe observable facts or what nature really is like? Well, it is believed to give us information about observed facts and their mutual dependence. There is an optimistic view, viz. that it gives us all the information obtainable about observable facts and their interdependence. But this view-which may or may not be correctis optimistic only inasmuch as it may flatter our pride to possess in principle all obtainable information. It is pessimistic in another respect, we might say epistemologically pessimistic. For the information we get as regards the causal dependence of observable facts is incomplete. (The cloven hoof must show up somewhere!) The gaps, eliminated from the wave picture, have withdrawn to the connection between the wave picture and the observable facts. The latter are not in one-to-one correspondence with the former. Plenty of ambiguity remains, and, as I said, some optimistic pessimists or pessimistic optimists believe that this ambiguity is essential, it cannot be helped.

This is the logical situation at present. I believe I have depicted it correctly, though I am quite aware that without examples the whole discussion has remained a little bloodless—just purely logical. I am also afraid that I have given you too unfavourable an impression of the wave theory of matter. I ought to amend both points. The wave theory is not of yester-

day and not of 25 years ago. It made its first appearance as the wave theory of light (Huygens 1690). For the better part of 100 years light waves were regarded as an incontrovertible reality, as something of which the real existence had been proved beyond all doubt by experiments on the diffraction and interference of light. I do not think that even today many physicists—certainly not experimentalists—are ready to endorse the statement



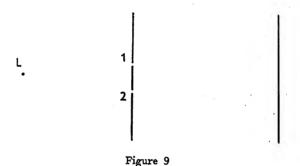
that 'light waves do not really exist, they are only waves of knowledge' (free quotation from Jeans).

If you observe a narrow luminous source L, a glowing Wollaston wire, a few thousandths of a millimetre thick, by a microscope whose objective lens is covered by a screen with a couple of parallel slits, you find (in the image plane conjugate to L) a system of coloured fringes which conform exactly and quantitively to the idea that light of a given colour is a wave motion of a certain small wave-

¹ Not the immediately following hundred years. Newton's authority eclipsed Huygens' theory for about a century.

length, shortest for violet, about twice as long for red light. This is one out of dozens of experiments that clinch the same view. Why, then, has this *reality* of the waves become doubtful? For two reasons:

(a) Similar experiments have been performed with beams of cathode rays (instead of light); and cathode rays—so it is said—manifestly consist of single electrons, which yield 'tracks' in the Wilson cloud chamber.



(b) There are reasons to assume that light itself also consists of single particles—called photons (from the Greek $\phi \tilde{\omega}_5 = \text{light}$).

Against this one may argue that nevertheless in both cases the concept of waves is unavoidable, if you wish to account for the interference fringes. And one may also argue that the particles are not identifiable objects, they might be regarded as explosion like events within the wave-front—just the events by which the wave-front manifests itself to observation. These events—so one might say—are to a certain

extent fortuitous, and that is why there is no strict causal connection between observations.

Let me explain in some detail why the phenomena, both in the case of light and in the case of cathode rays, cannot possibly be understood by the concept of single, individual, permanently existing corpuscles. This will also afford an example of what I call the 'gaps' in our description and of what I call the 'lack of individuality' of the particles. For the sake of

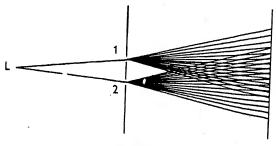


Figure 10

argument we simplify the experimental arrangement to the utmost. We consider a small, almost point-like source which emits corpuscles in all directions, and a screen with two small holes, with shutters, so that we can open first only the one, then only the other, then both. Behind the screen we have a photographic plate which collects the corpuscles that emerge from the openings. After the plate has been developed, it shows, let me assume, the marks of the single corpuscles that have hit it, each rendering a grain of silver-bromide developable,

so that it shows as a black speck after developing. (This is very near the truth.)

Now let us first open only one hole. You might expect that after exposing for some time we get a close cluster around one spot. This is not so. Apparently the particles are deflected from their straight

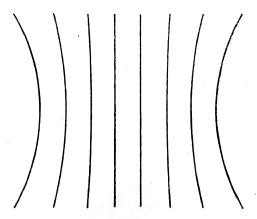


Figure 11

The lines indicate the places where there are few or no spots, while midway between any two lines the spots would be most frequent. The two straight lines in the middle are parallel to the slits.

path at the opening. You get a fairly wide spreading of black specks, though they are densest in the middle, becoming rarer at greater angles. If you open the second hole alone, you clearly get a similar pattern, only around a different centre.

Now let us open both holes at the same time and expose the plate just as long as before. What would

you expect-if the idea was correct, that single individual particles fly from the source to one of the holes, are deflected there, then continue along another straight line until they are caught by the plate? Clearly you would expect to get the two former patterns superposed. Thus in the region where the two fans overlap, if near a given point of the pattern you had, say, 25 spots per unit area in the first experiment and 16 more in the second, you would expect to find 25 + 16 = 41 in the third experiment. This is not so. Keeping to these numbers (and disregarding chance-fluctuations, for the sake of argument), you may find anything between 81 and only 1 spot, this depending on the precise place on the plate. It is decided by the difference of its distances from the holes. The result is that in the overlapping part we get dark fringes separated by fringes of scarcity.

(N.B. The numbers 1 and 81 are obtained as

$$(\sqrt{25} \pm \sqrt{16})^2 = (5 \pm 4)^2 = \frac{81}{1}$$
.)

If one wanted to keep up the idea of single individual particles flying continuously and independently either through one or through the other slit one would have to assume something quite ridiculous, namely that in some places on the plate the particles destroy each other to a large extent, while at other places they 'produce offspring'. This is not only ridiculous but can be refuted by experiment. (Making the source extremely weak and exposing for a very long time. This does not change the pattern!) The

only other alternative is to assume that a particle flying through the opening No. 1 is influenced also by the opening No. 2, and that in an extremely mysterious fashion.

We must, so it seems, give up the idea of tracing back to the source the history of a particle that manifests itself on the plate by reducing a grain of silver-bromide. We cannot tell where the particle was before it hit the plate. We cannot tell through which opening it has come. This is one of the typical gaps in the description of observable events, and very characteristic of the lack of individuality in the particle. We must think in terms of spherical waves emitted by the source, parts of each wave-front passing through both openings, and producing our interference pattern on the plate—but this pattern manifests itself to observation in the form of single particles.

THE ALLEGED BREAK-DOWN OF THE BARRIER BETWEEN SUBJECT AND OBJECT

It cannot be denied that the new physical aspect of nature of which I have tried to give you some idea by this example is very much more complicated than the old way which I called 'the classical ideal of uninterrupted, continuous description'. The very serious question arises naturally: Is this new and unfamiliar way of looking at things, which is at variance with the habits of everyday thinking—is it deeply rooted in the facts of observation, so that it

has come to stay and will never by got rid of again; or is this new aspect perhaps the mark, not of objective nature, but of the setting of the human mind, of the stage that our understanding of nature has reached at present?

This is an extremely difficult question to answer, because it is not even absolutely clear what this antithesis means: objective nature and human mind. For on the one hand I undoubtedly form part of nature, while on the other hand objective nature is known to me as a phenomenon of my mind only. Another point that we must keep in mind in pondering this question is this: that one is very easily deceived into regarding an acquired habit of thought as a peremptory postulate imposed by our mind on any theory of the physical world. The famous instance of this is Kant, who, as you know, termed space and time, as he knew them, the form of our mental intuition (Anschauung)—space being the form of external, time that of internal, intuition. Throughout the nineteenth century most philosophers followed him in this. I will not say that Kant's idea was completely wrong, but it was certainly too rigid and needed modification when new possibilities came to light, e.g. that space may be (and probably is) closed in itself, yet without boundaries; and that two events may happen in such a way that either of them may be regarded as the earlier one (this was the most amazing novel aspect in Einstein's 'Restricted' Theory of Relativity).

But let us return to our question, however poorly

it may be formulated: Is the impossibility of a continuous, gapless, uninterrupted description in space and time really founded in incontrovertible facts? The current opinion among physicists is, that this is the case. Bohr and Heisenberg have put forward a very ingenious theory about it, which is so easy to explain that it has entered most popular treatises on the subject—I should say, unfortunately; for its philosophical implication is usually misunderstood. I am going to argue against it, but first I must summarize it briefly.

It runs as follows. We cannot make any factual statement about a given natural object (or physical system) without 'getting in touch' with it. This 'touch' is a real physical interaction. Even if it consists only in 'looking at the object', the latter must be hit by light-rays and reflect them into the eye, or into some instrument of observation. This means that the object is interfered with by observing it. You cannot obtain any knowledge about an object while leaving it strictly isolated. The theory goes on to assert that this disturbance is neither irrelevant nor completely surveyable. Thus after any number of painstaking observations the object is left in a state of which some features (the last observed ones) are known, but others (those interfered with by the last observation) are not known, or not accurately known. This state of affairs is offered as the explanation why no complete, gapless description of a physical object is possible.

But obviously these inferences, even when granted,

tell me so far only that such a description cannot be actually accomplished, but they do not convince me that I should not be able to form in my mind a complete, gapless model, from which everything I can observe can be correctly inferred or foreseen, to the degree of certainty which the incompleteness of my observations allows. The situation might be such as in the beginning of a game of whist. By the rules of the game I can only have knowledge of one quarter of all the 52 cards. Still I know that each of the other players also has a certain lot of 13 cards, which will not change during the game; that nobody else can have a queen of hearts (because I have it); that there are exactly 6 clubs among the cards I do not know (because I happen to have 7)—and so on.

I say this interpretation suggests itself: that there is a fully determined physical object in existence, but I can never know all about it. However, this would be a complete misunderstanding of what Bohr and Heisenberg and those who follow them actually mean. They mean that the object has no existence independent of the observing subject. They mean that recent discoveries in physics have pushed forward to the mysterious boundary between the subject and the object, which thereby has turned out not to be a sharp boundary at all. We are to understand that we never observe an object without its being modified or tinged by our own activity in observing it. We are to understand that under the impact of our refined methods of observation, and of thinking

about the results of our experiments, that mysterious boundary between the subject and the object has broken down.

The opinion of what may be called our two foremost quantum theorists deserves, of course, careful attention; and the further fact that several other prominent scientists do not reject their opinion, but seem rather satisfied with it, adds to its claim to be thoroughly weighed. But in doing so, I cannot suppress certain objections.

I do not think I am prejudiced against the importance that science has from the purely human point of view. I had expressed by the original title of these lectures, and I have explained in the introductory passages, that I consider science an integrating part of our endeavour to answer the one great philosophical question which embraces all others, the one that Plotinus expressed by his brief: τίνες δὲ ἡμεῖς;—who are we? And more than that: I consider this not only one of the tasks, but the task, of science, the only one that really counts.

But with all that, I cannot believe (and this is my first objection)—I cannot believe that the deep philosophical enquiry into the relation between subject and object and into the true meaning of the distinction between them depends on the quantitative results of physical and chemical measurements with weighing scales, spectroscopes, microscopes, telescopes, with Geiger-Müller-counters, Wilson-chambers, photographic plates, arrangements for measuring the radioactive decay, and whatnot. It is

not very easy to say why I do not believe it. I feel a certain incongruity between the applied means and the problem to be solved. I do not feel quite so diffident with regard to other sciences, in particular biology, and quite especially genetics and the facts about evolution. But we shall not talk about this here and now.

On the other hand (and this is my second objection), the mere contention that every observation depends on both the subject and the object, which are inextricably interwoven—this contention is hardly new, it is almost as old as science itself. Though but scarce reports and quotations of the two great men from Abdera, Protagoras and Democritus, have come down to us across the twenty-four centuries that separate us from them, we can tell that they both in their way maintained that all our sensations, perceptions, and observations have a strong personal, subjective tinge and do not convey the nature of the thingin-itself (the difference between them was that Protagoras dispensed with the thing-in-itself, to him our sensations were the only truth, while Democritus thought differently). Since then the question has turned up whenever there was science; we might follow it through the centuries, speaking of Descartes', Leibnitz', Kant's attitudes towards it. We shall not do this. But I must mention one point, in order not to be accused of injustice towards the quantum physicists of our days. I said their statement that in perception and observation subject and object are inextricably interwoven is hardly new.

But they could make a case that something about it is new. I think it is true that in previous centuries, when discussing this question, one mostly had in mind two things, viz. (a) a direct physical impression caused by the object in the subject, and (b) the state of the subject that receives the impression. against this, in the present order of ideas the direct physical, causal, influence between the two is regarded as mutual. It is said that there is also an unavoidable and uncontrollable impression from the side of the subject onto the object. This aspect is new, and, I should say, more adequate anyhow. For physical action always is inter-action, it always is mutual. What remains doubtful to me is only just this: whether it is adequate to term one of the two physically interacting systems the 'subject'. For the observing mind is not a physical system, it cannot interact with any physical system. And it might be better to reserve the term 'subject' for the observing mind.

ATOMS OR QUANTA—THE COUNTER-SPELL OF OLD STANDING, TO ESCAPE THE INTRICACY OF THE CONTINUUM

Be this as it may, it seems worth our while to try to examine the matter from various angles. A point of view that I have previously touched on in these lectures and that does suggest itself is this, that our present difficulties in physical science are bound up with the notorious conceptional intricacy inherent

in the idea of the *continuum*. But this does not tell you much. How are they bound up? What precisely is the mutual relationship?

If you envisage the development of physics in the last half-century, you get the impression that the discontinuous aspect of nature has been forced upon us very much against our will. We seemed to feel quite happy with the continuum. Max Planck was seriously frightened by the idea of a discontinuous exchange of energy, which he had introduced (1900) in order to explain the distribution of energy in black-body-radiation. He made strong efforts to weaken the hypothesis, and, if possible, to get away from it, but in vain. Twenty-five years later the inventors of wave mechanics indulged for some time in the fond hope that they had paved the way of return to a classical continuous description, but again the hope was deceptive. Nature herself seemed to reject continuous description, and this refusal seemed to have nothing to do with the mathematicians' aporia in dealing with the continuum.

This is the impression you get from the last 50 years. But quantum theory dates 24 centuries further back, to Leucippus and Democritus. They invented the first discontinuity—isolated atoms embedded in empty space. Our notion of the elementary particle has historically descended from their notion of the atom and is conceptionally derived from their notion of the atom; we have simply held on to it. And these particles have now turned out to be quanta of energy, because—as Einstein discovered in 1905—

mass and energy are the same thing. So the idea of discontinuity is very old. How did it arise? I wish to establish that it originated precisely from the intricacy of the continuum, so to speak as a weapon in defence against it.

How did the ancient atomists come by the idea of atomism of matter? This question gains now a more than merely historical interest, it becomes epistemologically relevant. The question is sometimes asked in the following form-in a mood of utter amazement: How did those thinkers, with an extremely scanty knowledge of the laws of physics, indeed in complete ignorance of all the relevant experimental facts—how did they hit on the correct theory of the composition of material bodies? Occasionally you find people so bewildered by this 'lucky strike' that they actually declare it to be a chanceevent and refuse to give the ancient atomists any credit for it. They declare that their atomic theory has been a completely unfounded guess which might just as well have turned out a mistake. Needless to say, it is always a scientist, never a classical scholar, who reaches this strange conclusion.

I reject it. But then I must answer the question. That is not very difficult. The atomists and their ideas did not emerge suddenly out of nothing, they were preceded by the great development that began with Thales of Miletus (floruit 585 B.C.) more than a century earlier; they continue the awe-inspiring line of Ionian physiologoi. Their immediate predecessor in this line was Anaximenes, whose principal

doctrine consisted in underlining the all-importance of 'rarefaction and condensation'. From a careful consideration of everyday experience he abstracted the thesis that every piece of matter can take on the solid, the liquid, the gaseous and the 'fiery' state; that the changes between these states do not imply a change of nature, but are brought about geometrically, as it were, by the spreading of the same amount of matter over a larger and larger volume (rarefaction), or-in the opposite transitions-by its being reduced or compressed into a smaller and smaller volume. This idea is so absolutely to the point that a modern introduction into physical science could take it over without any relevant change. Moreover it is certainly not an unfounded guess, but the outcome of careful observation.

If you try to assimilate Anaximenes' idea, you naturally come to think that the change of properties of matter, say on rarefaction, must be caused by its parts receding at greater distances from each other. But it is extremely difficult to accomplish this in your imagination, if you think of matter as forming a gapless continuum. What should recede from what? The mathematicians of the same epoch considered a geometrical line as consisting of points. That is perhaps all right if you leave it alone. But if it is a material line and you begin to stretch it—would not its points recede from each other and leave gaps between them? For the stretching cannot produce new points and the same set of points cannot go to cover a greater interval.

From these difficulties, which reside in the mysterious character of the continuum, the easiest escape is the one taken by the atomists, namely to regard matter as consisting from the outset of isolated 'points' or rather of small particles, which recede from each other on rarefaction and approach to closer distances on condensation, while remaining themselves unchanged. The latter is an important by-product. Without it, the contention that in these processes matter stays intrinsically unchanged would remain very hazy. The atomist can tell what it means: the particles remain unchanged; only their geometrical constellation changes.

It would thus seem that physical science in its present form—in which it is the direct offspring, the uninterrupted continuation, of ancient science—was from its very beginning ushered in by the desire to avoid the haziness inherent in the conception of the continuum, the precarious side of which was then more felt than in modern times, until quite recently. Our helplessness vis-à-vis the continuum, reflected in the present difficulties of quantum theory, is not a late arrival, it stood godmother to the birth of science—an evil godmother, if you please, like the thirteenth fairy in the tale of the Sleeping Beauty. Her evil spell had for a long time been stemmed by the genial invention of atomism. This explains why atomism has proved so successful and durable and indispensable. It was not a happy guess by thinkers who 'really did not know anything about it'—it was the powerful counter-spell which naturally cannot

be dispensed with as long as the difficulty it is to exorcise survives.

By this I will not say that atomism will ever go by the board. Its invaluable findings—especially the statistical theory of heat—certainly never will. Nobody can tell the future. Atomism finds itself facing a serious crisis. Atoms—our modern atoms, the ultimate particles—must no longer be regarded as identifiable individuals. This is a stronger deviation from the original idea of an atom than anybody had ever contemplated. We must be prepared for anything.

WOULD PHYSICAL INDETERMINACY GIVE FREE WILL A CHANCE?

On p. 12 I briefly touched upon that old crux, the apparent contradiction between the deterministic view about material events and what is called in Latin liberum arbitrium indifferentiae, in modern language free will. I suppose you all know what I mean: since my mental life is obviously bound up very closely with the physiological goings on in my body, more especially in my brain, then, if the latter are strictly and uniquely determined by physical and chemical natural laws, what about my inalienable feeling that I take decisions to act in this or that way, what about my feeling responsibility for the decision I actually do take? Is not everything I do mechanically determined in advance by the material state of affairs in my brain, including modifications

caused by external bodies, and is not my feeling of liberty and responsibility deceptive?

This does strike us as a true aporia, which occurred for the first time to Democritus, who realised it fully—but left it alone; very wisely, I think. He fully realised it. While he adhered to his 'atoms and the void' as the only reasonable way of understanding objective nature, we have some definite utterances of his preserved, to the effect that he also realised that this whole picture of the atoms and the void was formed by the human mind on the evidence of sense perceptions, and nothing else; and other utterances where he states, almost in the words of Kant, that we know nothing about what any thing really is in itself, the ultimate truth remaining deeply in the dark.

Epicurus took over Democritus' physical theories (by the way, without acknowledgement); however, less wise, and very keen on conveying to his disciples a fair and sound and incontrovertible moral attitude, he tampered with physics and invented his famous (or ill-famed) swerves—strongly reminiscent of modern ideas about 'uncertainty' of physical events. I will not enter on details here; suffice it to say that he broke away from physical determinism in a rather childish way, which was not based on any experience and therefore had no consequences.

The problem itself never left us. It turned up very prominently—or at least a problem of closely similar *logical* structure turned up—with St Augustine of Hippo, as a theological *aporia*. The part of the Law of Nature is taken by the omniscient and

almighty God. But since to him who believes in God the Law of Nature is obviously His law, I think I am right in calling it very much the same problem.

As everybody knows, St Augustine's great difficulty was precisely this: God being omniscient and almighty, I cannot do a thing without His knowing and willing—not only consenting, but determining it. How, then, could I be responsible for it? I suppose the religious attitude to this form of the question eventually has to be that we are here confronted with a deep mystery into which we cannot penetrate, but which we certainly must not try to solve by denying responsibility. We must not try, I say; or we had better not try, for we fail pitiably. The feeling of responsibility is congenital, nobody can discard it.

But let us turn to the original form of the question and to the part physical determinism plays in it. Naturally the so-called 'crisis of causality' in the physics of our day seemed to raise strong hope of releasing us from this paradox or *aporia*.

Could perhaps the declared indeterminacy allow free will to step into the gap in the way that free will determines those events which the Law of Nature leaves undetermined? This hope is, at first sight, obvious and understandable.

In this crude form the attempt was made, and the idea, to a certain extent, worked out by the German physicist Pascual Jordan. I believe it to be both physically and morally an impossible solution. As regards the first: according to our present view the

quantum laws, though they leave the single event undetermined, predict a quite definite statistics of events when the same situation occurs again and again. If these statistics are interfered with by any agent, this agent violates the laws of quantum mechanics just as objectionably as if it interfered -in pre-quantum physics-with a strictly causal mechanical law. Now we know that there is no statistics in the reaction of the same person to precisely the same moral situation—the rule is that the same individual in the same situation acts again precisely in the same manner. (Mind you, in precisely the same situation; this does not mean that a criminal or addict cannot be converted or healed by persuasion and example or whatnot-by strong external influence; but this, of course, means that the situation is changed.) The inference is that Jordan's assumption—the direct stepping in of free will to fill the gap of indeterminacy—does amount to an interference with the laws of nature, even in their form accepted in quantum theory. But at that price, of course, we can have everything. This is not a solution of the dilemma.

The moral objection was strongly emphasized by the German philosopher Ernst Cassirer (who died in 1945 in New York as an exile from Nazi Germany). Cassirer's extended criticism of Jordan's ideas is based on a thorough familiarity with the situation in physics. I shall try to summarize it briefly; I would say, it amounts to this. Free will in man includes as its most relevant part man's ethical behaviour. Sup-

posing the physical events in space and time actually are to a large extent not strictly determined but subject to pure chance, as most physicists in our time believe, then this haphazard side of the goings-on in the material world is certainly (says Cassirer) the very last to be invoked as the physical correlate of man's ethical behaviour. For this is anything but haphazard, it is intensely determined by motives ranging from the lowest to the most sublime sort, from greed and spite to genuine love of the fellow creature or sincere religious devotion. Cassirer's lucid discussion makes one feel so strongly the absurdity of basing free will, including ethics, on physical haphazard that the previous difficulty, the antagonism between free will and determinism, dwindles and almost vanishes under the mighty blows Cassirer deals to the opposite view. Even the reduced extent of predictability' (Cassirer adds) 'still granted by Quantum Mechanics, would amply suffice to destroy ethical freedom, if the concept and true meaning of the latter were irreconcilable with predictability'. Indeed, one begins to wonder whether the supposed paradox is really so shocking, and whether physical determinism is not perhaps quite a suitable correlate to the mental phenomenon of will, which is not always easy to predict 'from outside', but usually extremely determined 'from inside'. To my mind this is the most valuable outcome of the whole controversy: the scale is turned in favour of a possible reconciliation of free will with physical determinism, when we realise how inadequate a basis physical

haphazard provides for ethics. One could enlarge on this point. Innumerable passages could be adduced from poets and novelists to clinch it. In John Galsworthy's novel The Dark Flower (Part I, 13, second paragraph) the scattered thoughts of a young lad at night hit on this: 'But that was it—you never could think what things would be like if they weren't just what and where they were. You never knew what was coming, either; and yet, when it came, it seemed as if nothing else ever could have come. That was queer—you could do anything you liked until you'd done it, but when you had done it then you knew, of course, that you must always have had to . . .' There is a famous passage in Wallenstein's Tod (II.3):

Des Menschen Taten und Gedanken, wisst! Sind nicht wie Meeres blindbewegte Wellen. Die innre Welt, sein Mikrokosmus, ist Der tiefe Schacht, aus dem sie ewig quellen. Sie sind notwendig, wie des Baumes Frucht; Sie kann der Zufall gaukelnd nicht verwandeln. Hab' ich des Menschen Kern erst untersucht, So weiss ich auch sein Wollen und sein Handeln.

Be ye aware: man's thinking and man's deeds
Are not like the ocean's blindly surging spray.
His inner world, his microcosmus, feeds
The profound shaft from which they pour to the day.
They are needful as its fruit is in a tree,
Unalterable by blindly juggling chance.
Once into a man's deep core I probing see,
His will and act I'll tell you in advance.

It is true that in their context these lines refer to

Wallenstein's devout belief in astrology, which we are not inclined to share. But is not the very lure of astrology, the irresistible attraction it has for scores of centuries exerted on men's minds, witness to the fact that we are not prepared to regard our fate as the outcome of pure chance, even though, or rather just because, it largely depends on our taking the right decision in the right moment? (We usually lack the full information needed for this purpose; and that is where astrology comes in!)

THE BAR TO PREDICTION, ACCORDING TO NIELS BOHR

But let us return to our subject proper. A much more serious and interesting attempt to explain the difficulty away was founded by Bohr and Heisenberg on the idea, mentioned above, that there is an unavoidable and uncontrollable mutual interaction between the observer and the observed physical object. Their ratiocination is briefly as follows. The alleged paradox consists in this, that according to the mechanistic view, by procuring an exact knowledge of the configuration and velocities of all the elementary particles in a man's body, including his brain, one could predict his voluntary actionswhich thereby cease to be what he believes them to be, namely voluntary. The fact that we cannot actually procure this detailed knowledge is no great help. Even the theoretical predictability shocks us.

To this Bohr answers that the knowledge cannot

even be procured in principle, not even in theory, because such accurate observation would involve so strong an interference with 'the object' (the man's body) as to dissociate it into single particles—in fact kill him so efficiently that not even a corpse would be left for burial. At any rate, no prediction of behaviour would result, before the 'object' is far beyond the state of exhibiting any voluntary behaviour.

The emphasis is of course on the phrase 'in principle'. That the said knowledge cannot actually be procured, not even for the simplest living organism, let alone a higher animal like man, is clear also without quantum theory and uncertainty relation.

Bohr's consideration is no doubt interesting. Yet, I should say, we are more convicted by it than convinced, as in some mathematical proofs: you must grant A and B, then follows C and D, and so on, you cannot object to any single step; finally follows the interesting result Z. You have to accept it, but you cannot see how it really comes about, the proof gives no hint of that. In the present case I would say: Bohr's considerations show you that the present views in physics-mainly on account of the lack of strict causality (or on account of the uncertainty relation) —bar the objectionable predictability in principle. But you cannot see how this comes about. In view of the close relation Bohr's reasoning has to the lack of observable strict causality, you even incline to suspect that it is only Jordan's suggestion over again, but in a more careful disguise, so as to be shielded from Cassirer's arguments.

One can make a case for this being so. Indeed, I think I must accuse Bohr-though in actual fact he is one of the kindliest persons I ever came to know of an unnecessary cruelty for his proposing to kill his victim by observation. I cannot see what purpose it should serve. It will never, according to quantum mechanics, yield us the full set of configuration and velocities of all the particles, because according to our present views this is impossible. The equivalent of this complete knowledge in classical physics is in quantum physics a so-called maximum observation, which yields the maximum knowledge that can be obtained, nay, that has any meaning. Nothing in the views accepted at present precludes that we should obtain this maximum knowledge of a living body. We must admit the possibility in principle, even though we know perfectly well that practically it cannot be achieved. This state of affairs is exactly the same as with complete knowledge in classical physics. Furthermore, precisely as in classical physics, you can from a maximum observation, yielding maximum knowledge now, deduce, in principle, maximum knowledge at any later time. (You must, of course, procure maximum knowledge also about all agents that act on your object in the meantime; but that is, in principle, possible and is again absolutely analogous to the case of classical mechanistic physics.) The fundamental difference is only this, that the said maximum knowledge at that later time may leave you in doubt about very conspicuous features of the actual observable behaviour of your object at

that later time—the more so, the longer the time that has elapsed.

It would thus appear that Bohr's considerations adduce a *physical* unpredictability of the behaviour of a living body again precisely from the lack of strict causation, maintained by quantum theory. Whether or no this physical indeterminacy plays any relevant role in organic life, we must, I think, sternly refuse to make it the physical counterpart of voluntary actions of living beings, for the reasons outlined before.

The net result is that quantum physics has nothing to do with the free-will problem. If there is such a problem, it is not furthered a whit by the latest development in physics. To quote Ernst Cassirer again: 'Thus it is clear . . . that a possible change in the physical concept of causality can have no immediate bearing on ethics'.



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